

# Dynamical Casimir effect at finite temperature

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## Abstract

Thermal effects on the creation of particles under the influence of time-dependent boundary conditions are investigated. The dominant temperature correction to the energy radiated by a moving mirror is derived by means of response theory. For a resonantly vibrating cavity the thermal effect on the number of created photons is obtained non-perturbatively. Finite temperatures can enhance the pure vacuum effect by several orders of magnitude. The relevance of finite temperature effects for the experimental verification of the dynamical Casimir effect is addressed.

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In 1948 H. B. G. Casimir predicted an attractive force between two perfectly conducting, parallel plates placed in the vacuum [1]. A variety of fundamental and measurable consequences of quantum fluctuations under the influence of external conditions have been derived during the last decades (see e.g. [2] for reviews). With the recent precision measurement of the Casimir force performed by Mohideen *et al.* [3] and by Lamoureaux [4] conclusive tests of Casimir's prediction are now available. They confirm the basic concepts of quantum field theory in the presence of static external constraints.

One could expect that the success in measuring the static Casimir force could also intensify experimental efforts in order to verify a not less fundamental prediction, namely the *dynamical Casimir effect*, i.e., the creation of particles out of the vacuum induced by the interaction with dynamical external constraints. The phenomenon of vacuum radiation induced by moving mirrors [5] initiated intensive studies (see e.g. [6] and references therein). In particular the creation of photons in vibrating cavities seems to be the most promising scenario for a possible experimental verification of motion-induced vacuum radiation (see e.g. [7–11]).

The thermodynamics of the static Casimir effect has been investigated intensively (see e.g. [13] and references therein). In that context temperature effects are known to even dominate the pure vacuum effect (at  $T = 0$ ) and, in consequence, have to be taken into account when analysing the data in measurements of the static Casimir force. In contrast to this the dynamical Casimir effect at finite temperature so far has not been subject of research. It has been anticipated in a recent investigation by Lambrecht *et al.* [12] that temperature effects could play an important role for the generation of a photon pulse in a vibrating Fabry-Pérot cavity. However, realistic calculations of thermal effects on quantum radiation within the framework of quantum field theory of time-dependent systems at finite temperature are not yet available.

Accordingly, it is our major intention to close this gap and to provide a generalization of the Hamiltonian approach presented recently in Refs. [14,15]. In this letter we focus the discussion predominantly on results obtained for the thermal contribution to photon production in a resonantly vibrating cavity as one of the most relevant configuration when aiming for experimental tests of the dynamical Casimir effect [9]. We like to address the question whether or not the effect of quantum radiation might be covered by the thermal background and we will examine the conditions under which it remains most significant even at large temperatures. Details of the derivations and further applications of the formalism will be presented in a forthcoming publication [16].

We recall the canonical formalism in Ref. [14], where we considered a constrained, non-interacting, real, massless scalar field. Similarly, the formalism also holds for bosonic quantum fields interacting with classical external background fields [15]. The boundary respectively the background may undergo small but arbitrary dynamical changes resulting in an additional interaction Hamiltonian  $\hat{H}_I$  which is assumed to be switched on and off at asymptotic times  $t \rightarrow -\infty$  and  $t \rightarrow +\infty$ , respectively. For the boundary being initially at rest the (closed) system consisting of the scalar field enclosed by the boundary is assumed to be at thermal equilibrium described by a statistical operator  $\hat{\rho}(t \rightarrow -\infty) = \hat{\rho}_0$ . The existence of a total Hamiltonian  $\hat{H} = \hat{H}_0 + \hat{H}_I$  describing the evolution of the system implies that any kind of backreaction of the quantum field upon the dynamics of the external constraints or any relaxation processes will be neglected. In addition, there should be no measurements on the quantum systems during the dynamical phase. We adopt the interaction representation. When the boundary or the background field experiences dynamical changes the system will no longer remain at thermal equilibrium. The time-evolution of the statistical operator  $\hat{\rho}$  is governed by the time-dependent interaction Hamiltonian  $\hat{H}_I$  resulting in the von Neumann equation (units where  $\hbar = c = k_B = 1$  are used throughout):

$$i \frac{d\hat{\rho}}{dt} = [\hat{H}_I, \hat{\rho}] \quad , \quad (1)$$

together with the initial condition  $\hat{\rho}(t \rightarrow -\infty) = \hat{\rho}_0$ . Here we disregard an explicit time-dependence  $(\partial\hat{\rho}/\partial t)_{\text{exp}}$  that could account for relaxation or backreaction processes. Accordingly, this equation of motion can be integrated formally with the aid of the time-evolution operator

$$\hat{U}(t', t) = \mathcal{T} \left[ \exp \left( -i \int_t^{t'} dt_1 \hat{H}_I(t_1) \right) \right] \quad , \quad (2)$$

where  $\mathcal{T}$  denotes time-ordering. The thermal expectation value at asymptotic times  $t \rightarrow \infty$  of any relevant observable  $\hat{A}$  is determined by

$$\langle \hat{A} \rangle = \text{Tr} \{ \hat{A} \hat{\rho}(t \rightarrow \infty) \} = \text{Tr} \{ \hat{A} \hat{U} \hat{\rho}_0 \hat{U}^\dagger \} \quad . \quad (3)$$

The trace  $\text{Tr}\{\dots\}$  involved is taken most conveniently over the Fock space of the unperturbed Hamiltonian  $\hat{H}_0$  referring to the initial ensemble (with  $\text{Tr}\{\hat{\rho}_0\} = 1$ ). The microscopic entropy  $S = -\text{Tr} \{ \hat{\rho} \ln \hat{\rho} \} = -\text{Tr} \{ \hat{\rho}_0 \ln \hat{\rho}_0 \} = S_0$  remains constant in time. A more detailed examination of the thermodynamical aspects involved (e.g. effective entropy) will be given in [16]. Eqs. (1) – (3) generalize the canonical approach to quantum radiation of Ref. [14,15] to account for finite-temperature effects as well. In order to explore the influence of finite temperatures on the dynamical Casimir effect we have to investigate the thermal expectation value of the number operator  $\hat{N}_\lambda = \hat{a}_\lambda^\dagger \hat{a}_\lambda$  of particles with frequency  $\Omega_\lambda^0$ . Note, that the proper definition of (quasi-) particle creation- (annihilation-) operators  $\hat{a}_\lambda^\dagger$  ( $\hat{a}_\lambda$ ) is provided with respect to the ground state  $|0\rangle$  of the unperturbed Hamiltonian  $\hat{H}_0$ , which becomes diagonal, i.e.,  $\hat{H}_0 = \Omega_\lambda^0 (\hat{N}_\lambda + 1/2)$ . The total radiated energy associated with the dynamical Casimir-effect can be deduced from the expectation value Eq. (3) of the normal-ordered unperturbed Hamiltonian  $\text{Tr}\{:\hat{H}_0: \hat{\rho}\}$ .

In the following we consider the generic case of a constrained massless scalar or vector field at finite temperature  $T = 1/\beta$  described initially ( $t \rightarrow -\infty$ ) by the statistical operator (canonical ensemble)

$$\hat{\rho}_0 = \exp(-\beta \hat{H}_0) / Z_0 \quad . \quad (4)$$

The closed system is assumed to be initially at thermal equilibrium. The system leaves the thermal equilibrium when undergoing some dynamical changes. Eq. (3) allows for a systematic perturbative approach in those cases, for which a closed expression for the time-evolution operator is not available. Especially for the thermal expectation value of the

spectral number density  $\langle \hat{N}_\lambda \rangle$  one obtains up to quadratic response neglecting corrections of order  $\mathcal{O}(\hat{H}_I^3)$

$$\begin{aligned} \langle \hat{N}_\lambda \rangle &= \text{Tr} \{ \hat{N}_\lambda \hat{\rho}_0 \} + \text{Tr} \left\{ \hat{N}_\lambda \left[ \int dt \hat{H}_I(t), \hat{\rho}_0 \right] \int dt \hat{H}_I(t) \right\} \\ &= \langle \hat{N}_\lambda \rangle_0 + \Delta N_\lambda \quad . \end{aligned} \quad (5)$$

The first term  $\langle \hat{N}_\lambda \rangle_0 = \text{Tr} \{ \hat{N}_\lambda \hat{\rho}_0 \}$  represents the usual thermal Bose-distribution function while the second term  $\Delta N_\lambda$  denotes the motion-induced change of the number of particles (e.g. photons) with frequency  $\Omega_\lambda^0$  at a given temperature  $T$ . The total radiated energy, i.e. the change of the internal energy associated with quantum radiation is determined by  $\Delta E = \Omega_\lambda^0 \Delta N_\lambda$  and can be deduced according to  $E = \langle : \hat{H}_0 : \rangle = \langle : \hat{H}_0 : \rangle_0 + \Delta E$ .

Let us now turn to the generic particle-creation processes. In the case of constrained bosonic quantum fields satisfying linear equations of motion the general form of the self-adjoint perturbation Hamiltonian induced by the interaction with the external conditions assumes the rather general form (sum convention)

$$\int dt \hat{H}_I(t) = \frac{1}{2} \left( \mathcal{S}_{\mu\nu} \hat{a}_\mu^+ \hat{a}_\nu^+ + \mathcal{S}_{\mu\nu}^* \hat{a}_\mu \hat{a}_\nu \right) + \mathcal{U}_{\mu\nu} \hat{a}_\mu^+ \hat{a}_\nu + \mathcal{C} \quad (6)$$

with  $\mathcal{S}_{\mu\nu} = \mathcal{S}_{\nu\mu}$  and  $\mathcal{U}_{\mu\nu} = \mathcal{U}_{\nu\mu}^*$ . The constant  $\mathcal{C}$  gives rise to a pure phase factor and thus drops out in any expectation value (3). The  $\mathcal{S}$ -term may be interpreted as a generator of a multi-mode squeezing operator and the  $\mathcal{U}$ -term may be envisaged as a hopping operator. Evaluating the quadratic response according to Eq. (5) the change of the number of particles yields

$$\begin{aligned} \Delta N_\lambda &= |\mathcal{S}_{\lambda\rho}|^2 \left( 1 + \langle \hat{N}_\rho \rangle_0 + \langle \hat{N}_\lambda \rangle_0 \right) \\ &\quad + |\mathcal{U}_{\lambda\rho}|^2 \left( \langle \hat{N}_\rho \rangle_0 - \langle \hat{N}_\lambda \rangle_0 \right) \quad . \end{aligned} \quad (7)$$

Note, that the  $\mathcal{S}$ -term contains the pure vacuum contribution  $\Delta N_\lambda(\text{vac}) = |\mathcal{S}_{\lambda\rho}|^2$ . One observes that only the  $\mathcal{S}$ -term contributes to the total number of created particles (sum over  $\lambda$ ), while the  $\mathcal{U}$ -term does not increase the total number of particles (similar terms appear in master-equations). However, it modifies the configurations of occupied particle states within the ensemble and thus also increases the total energy.

For trembling cavities the perturbation Hamiltonian (cf. Ref. [14] for details)

$$\hat{H}_I(t) = \hat{q}_\mu^2(t) \Delta\Omega_\mu^2(t)/2 + \hat{q}_\mu \hat{p}_\nu \mathcal{M}_{\mu\nu}(t) \quad (8)$$

appears as a sum of a squeezing (first term) and a velocity term (second term). The squeezing term involves the deviations  $\Delta\Omega_\mu^2$  of the eigenmode spectrum induced by a change of the

shape of the cavity, while the velocity contribution involves the (antisymmetric) intermode couplings  $\mathcal{M}_{\mu\nu}$  arising from the motion of the boundary.

In scenarios, where the velocity effect is supposed to be negligible the matrix  $\mathcal{S}$  becomes diagonal and the  $\mathcal{U}$ -term does not contribute at all. In view of Eq. (7) we identify the corresponding particle-production rate as the product of the pure vacuum-squeezing effect  $\Delta N_\lambda^S(\text{vac}) = |\mathcal{S}_{\lambda\lambda}|^2$  times a thermal distribution factor, i.e.  $\Delta N_\lambda^S = \Delta N_\lambda^S(\text{vac}) (1 + 2\langle \hat{N}_\lambda \rangle_0)$ . It gives rise to a linear dependence on  $T$  in the high-temperature limit.

The pure velocity effect may be illustrated most simply by considering a single moving mirror in 1+1 dimensions. For a single mirror placed at a position  $\eta(t)$  the discrete eigenmodes  $\Omega_\lambda^0$  have to be replaced by the continuous variable  $k$ . The total radiated energy is derived as

$$E = \frac{1}{12\pi} \int dt \ddot{\eta}^2(t) + \frac{\pi}{3} T^2 \int dt \dot{\eta}^2(t) \quad , \quad (9)$$

which generalizes the zero-temperature result (first term) obtained by Fulling and Davis and by Ford and Vilenkin [5]. The ratio of the finite temperature correction to the radiated energy and the pure vacuum contribution turns out to be of the order  $\mathcal{O}(T^2\tau^2)$ , where  $\tau$  denotes a characteristic time scale of the underlying dynamics.

Let us now investigate the finite-temperature effects on the dynamical Casimir-effect in a resonantly vibrating cavity. In order to allow for an experimental verification the number of motion-induced created particles should be as large as possible. One way to achieve this goal is to utilize the phenomenon of parametric resonance, which occurs in the case of harmonically time-dependent perturbations characterized by some frequency  $\omega$ . Obtaining large numbers may indicate that one has left the region, where second-order perturbation theory does apply.

In the case of oscillating disturbances, however, it is possible to evaluate the time-evolution operator to all orders of  $\hat{H}_I$  analytically employing yet another approximation, the so-called rotating wave approximation (see e.g. [7]). Let us assume that the explicit time-dependence of the perturbation Hamiltonian possesses an oscillatory behaviour like  $\varepsilon \sin(2\omega t)$  during a sufficiently long period of time  $\mathsf{T}$ , such that the conditions  $\omega\mathsf{T} \gg 1$ ,  $\varepsilon \ll 1$  and  $\varepsilon\omega\mathsf{T} = \mathcal{O}(1)$  hold. The main consequence of the rotating wave approximation consists in the fact that it allows for the derivation of a time-independent, effective Hamiltonian  $\hat{H}_\omega^{\text{eff}}$  after performing the integration over time according to Eq. (6):

$$\int_0^\mathsf{T} dt \hat{H}_I \approx \mathsf{T} \hat{H}_\omega^{\text{eff}} \quad . \quad (10)$$

This effective Hamiltonian accounts essentially for the dominant contribution arising from the frequency  $\omega$ . For the evolution operator Eq. (2) this approximation implies the neglect of all time-ordering effects (all oscillating terms involving commutators average out).

For a vibrating cavity the effective Hamiltonian  $\hat{H}_\omega^{\text{eff}}$  is easily calculated from the interaction operator (8). Assuming a harmonic time-dependence proportional to  $\varepsilon \sin(2\omega t)$  or  $\varepsilon \cos(2\omega t)$  for both, the squeezing ( $\Delta\Omega_\mu^2$ ) and the velocity terms ( $\mathcal{M}_{\mu\nu}$ ) only those terms will survive, which match the resonance conditions. For the squeezing term the resonance condition reads  $\Omega_\lambda^0 = \omega$  and for the velocity term  $|\Omega_\mu^0 \pm \Omega_\nu^0| = 2\omega$ , respectively. Accordingly, the squeezing effect creates always particles with the frequency  $\omega$  provided this cavity mode does exist. We restrict our further consideration to the situation, where the oscillation frequency  $\omega$  corresponds to the lowest cavity mode  $\omega = \min\{\Omega_\lambda^0\} = \Omega_1^0$ . The fundamental resonance frequency is determined by the characteristic size  $L$  of the cavity, e.g.  $\Omega_1^0 = \sqrt{3}\pi/L$  for a cubic cavity. For the lowest mode the resonance condition for squeezing  $\Omega_\lambda^0 = \Omega_1^0$  is satisfied automatically. Whether the resonance condition for the velocity effect  $|\Omega_\mu^0 \pm \Omega_\nu^0| = 2\Omega_1^0$  can be satisfied or not depends on the spectrum of the particular cavity under consideration. For a one-dimensional cavity the eigenvalues  $\Omega_\lambda^0$  are proportional to integers and thus it can be satisfied leading to an additional velocity contribution. For most cases of higher-dimensional cavities, e.g. a cubic one, this condition cannot be fulfilled. Thus the velocity effect does not contribute within the rotating wave approximation (cf. Ref. [9]).

In consequence, only the squeezing term contributes. The effective Hamiltonian can be derived immediately for the lowest cavity mode. From the contributing  $\Delta\Omega_\lambda$ -terms one obtains  $\hat{H}_1^{\text{eff}} = i\Omega_1^0\varepsilon/4 \left[ (\hat{a}_1^\dagger)^2 - (\hat{a}_1)^2 \right]$  respectively for the time-evolution operator

$$\hat{U}(\mathbb{T}, 0) \approx \exp \left\{ \frac{\Omega_1^0\varepsilon\mathbb{T}}{4} \left[ (\hat{a}_1^\dagger)^2 - (\hat{a}_1)^2 \right] \right\} = \hat{S}_1 \quad . \quad (11)$$

Note, that within this approximation  $\hat{U}$  coincides with a squeezing operator  $\hat{S}_1$  for the lowest mode  $\lambda = 1$ . This confirms the notion of the  $\Delta\Omega_\mu^2$ -terms in (8) as squeezing contribution. After having derived a closed expression for the time-evolution operator this enables us to calculate the expectation value for the number operator to all orders in  $\hat{H}_1^{\text{eff}}$

$$\langle \hat{N}_\lambda \rangle \approx \langle \hat{N}_\lambda \rangle_0 + \delta_{\lambda 1} \sinh^2 \left( \frac{\varepsilon\Omega_1^0\mathbb{T}}{2} \right) \left( 1 + 2\langle \hat{N}_1 \rangle_0 \right) \quad . \quad (12)$$

This non-perturbative result implies that at finite temperature the number of photons  $\Delta N_1$  created resonantly in the lowest cavity mode increases exponentially. The vacuum creation rate  $\Delta N_1^{\text{S}}(\text{vac}) = \sinh^2(\varepsilon\Omega_1^0\mathbb{T}/2)$  (see Ref. [9]) gets enhanced by a thermal distribution factor.

Equation (12) represents one essential result of our investigations. In order to indicate its experimental relevance one may specify the characteristic parameters. In Fig. 1 the total number of photons in the fundamental mode  $\omega = \Omega_1^0$  inside of a cavity of a characteristic size  $L$  is depicted as a function of the time-duration  $\mathsf{T}$  of the vibration of its wall. The contribution  $\Delta N_1$  of additional, motion-induced photon creation at  $T = 0$ , i.e. the pure vacuum effect (dashed curve) is compared with the corresponding contribution at a temperature  $T$  (solid curve). The shadowed area indicates the number of photons  $\langle \hat{N}_1 \rangle_0$  present at the temperature  $T$  before the system will undergo dynamical changes together with its thermal variance  $\sigma_0(N_\lambda) = \sqrt{\langle \hat{N}_\lambda^2 \rangle_0 - \langle \hat{N}_\lambda \rangle_0^2} \approx \langle \hat{N}_\lambda \rangle_0$  for  $\langle \hat{N}_\lambda \rangle_0 \gg 1$ . The latter reflects the uncertainty when measuring the number of photons at a given temperature  $T$ . For room temperature  $T \approx 290$  K, which corresponds to thermal wave lengths of about  $50 \mu\text{m}$  and considering a cavity of a typical size  $L \approx 1$  cm one obtains a thermal factor  $(1 + 2\langle \hat{N}_1 \rangle_0)$  of order  $10^3$ . As a consequence after the vibration time  $\mathsf{T}$  the number of photons  $\Delta N_1$  created by the dynamical Casimir effect at the given temperature will be three orders of magnitude larger in comparison with the pure vacuum effect at  $T = 0$ . At finite temperatures the dynamical Casimir effect should become observable even after shorter vibration times  $\mathsf{T}$  as one would expect from looking at the pure vacuum effect. This strong enhancement of the dynamical Casimir effect occurring at finite temperatures could be exploited in experiments to verify the phenomenon of quantum radiation as long as backreaction processes can be neglected. Of course, one has to ensure conditions that will lead to a significant vacuum effect as well. This implies that the argument of the hyperbolic sine function in Eq. (11), i.e. the squeezing parameter  $\varepsilon\omega\mathsf{T}/2$  should be at least of order 1. An estimate of the maximal value of the dimensionless amplitude of the resonance wall vibration  $\varepsilon_{\text{max}} < 10^{-8}$  is given in Ref. [9]. It still remains as a challenge whether or not the requirement  $\varepsilon\omega\mathsf{T}/2 \approx 1$  could be achieved in a realistic experiment. We conclude: Provided an experimental device for generating a considerable vacuum contribution becomes feasible, then we predict a strong enhancement of the dynamical Casimir effect at finite temperature. From the theoretical point of view there is no definite need to perform an even more involved experiment at low temperatures.

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## REFERENCES

- [1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).
- [2] G. Plunien, B. Müller, and W. Greiner, Phys. Rep. **134**, 87 (1986); V. M. Mostepaneko and N. N. Turnov, *The Casimir Effect and its Applications*, (Claredon, London, 1997); M. Bordag, *The Casimir Effect 50 years later* (World Scientific, Singapore, 1999).
- [3] U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549 (1998).
- [4] S. K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997).
- [5] G. Moore, J. Math. Phys. **11**, 2679 (1970); S. A. Fulling and P. C. W. Davis, Proc. R. Soc. London, Ser. A **348**, (1976); L. H. Ford and A. Vilenkin, Phys. Rev. D **25**, 2569 (1982).
- [6] G. Barton and C. Eberlein, Ann. Phys. (N.Y.) **227**, 222 (1993).
- [7] C. K. Law, Phys. Rev. Lett. **73**, 1931 (1994); C. K. Law, Phys. Rev. A **49**, 433 (1994).
- [8] V. V. Dodonov, Phys. Lett. A **207**, 126 (1995).
- [9] V. V. Dodonov and A. B. Klimov, Phys. Rev. A **53**, 2664 (1996).
- [10] O. Méplan and C. Gignaux, Phys. Rev. Lett. **76**, 408 (1996).
- [11] A. Lambrecht, M.-T. Jaekel, and S. Reynaud, Phys. Rev. Lett. **77**, 615 (1996).
- [12] A. Lambrecht, M.-T. Jaekel, and S. Reynaud, Europhys. Lett. **43**, 147 (1998).
- [13] H. Mitter and D. Robaschik, *Thermodynamics of the Casimir effect*, e-preprint: quant-ph/9902074; G. Plunien, B. Müller, and W. Greiner, Physica A **145**, 202 (1987).
- [14] R. Schützhold, G. Plunien, and G. Soff, Phys. Rev. A **57**, 2311 (1998).
- [15] R. Schützhold, G. Plunien, and G. Soff, Phys. Rev. A **58**, 1783 (1998).
- [16] R. Schützhold and G. Plunien, *Quantum radiation at finite temperature*, in preparation



# FIGURES

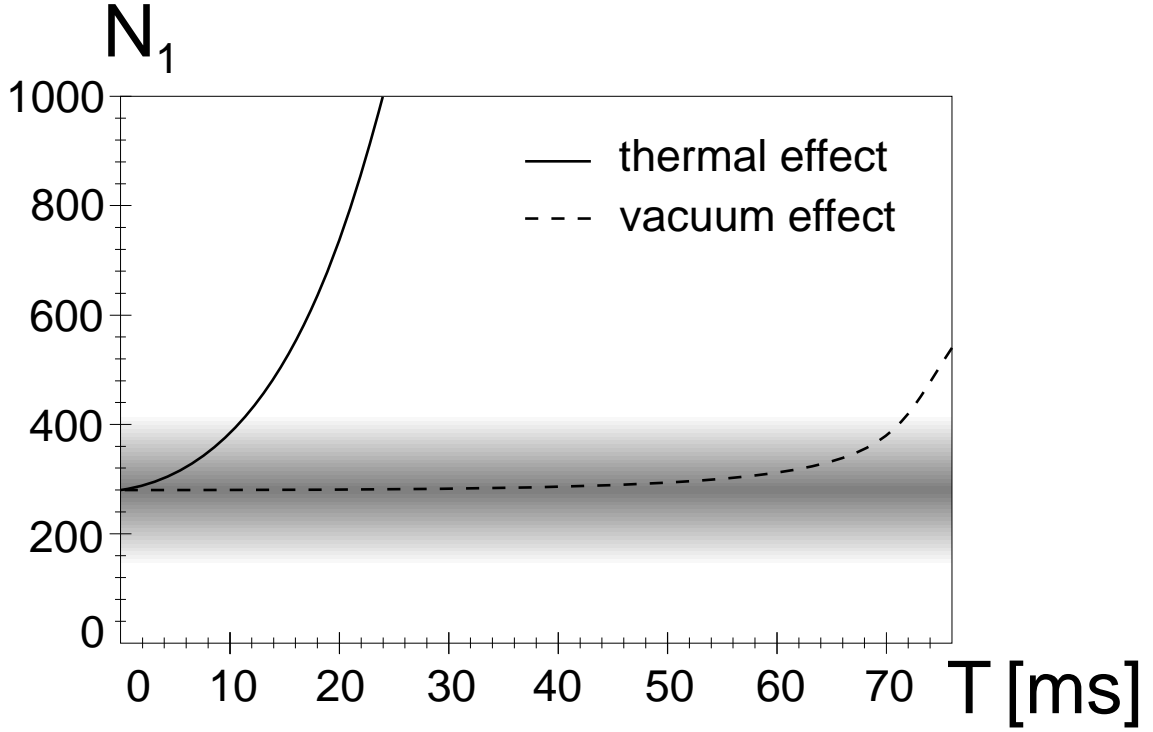


FIG. 1. Number of photons  $N_1$  produced via the dynamical Casimir effect inside a resonantly vibrating cavity for a fixed value of the squeezing parameter  $\varepsilon\omega/2$  as a function of the time duration  $T$ . The fundamental frequency is chosen to be  $\omega = 146$  GHz corresponding to a typical size of the cavity of 1 cm. For the dimensionless amplitude a value of  $\varepsilon = 6 \cdot 10^{-10}$  has been assumed. A finite temperature ( $T = 290$  K) can enhance the effect (solid line) by several orders of magnitudes compared with the pure vacuum effect at  $T = 0$  (dashed line). The thermal background is indicated by the shadowed area (half of the thermal variance  $\sigma_0$  of the photon number  $N_1$ ).